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A SUPERREFLEXIVE BANACH SPACE WITH A FINITE DIMENSIONAL DECOMPOSITION SO THAT NO LARGE SUBSPACE HAS A BASIS

BY

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ABSTRACT

The main result of the present paper is the construction of a Banach space with finite dimensional decomposition in which no large subspace has a basis. This answers a question raised by P. Casazza. The paper also contains various results on subspaces of direct sums of spaces and an investigation of spaces closely related to spaces constructed by the first named author.

Introduction

In [11], Pełczyński proved that if X is a separable Banach space with the bounded approximation property, then there is a Banach space E so that $X \oplus E$ has a basis. This result was actually discovered independently by Johnson, Rosenthal and Zippin [4]. Casazza [3] proved recently that if X is a separable Banach space with the commuting bounded approximation property, then there is a subspace F of X so that both F and $X \oplus F$ have finite dimensional decompositions. His construction has the advantage that many of the properties of X are inherited by the subspace $X \oplus F$ of $X \oplus X$. He then asks if the space E in Pełczyński's result can be chosen to be a subspace of X. Here we answer this question in the negative. We shall construct a superreflexive Banach space X with a finite dimensional decomposition so that if F is a subspace of $X \oplus X$ containing $X \oplus \{0\}$, then F does not have a basis. This is equivalent to saying that for no subspace E of X does $X \oplus E$ have a basis.

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The construction of the space X is essentially the same as in Szarek [13], [14]; however the technique of the proofs is based on the methods developed in [8] and [9]. The credit for the first use of such methods to construct pathological infinite dimensional Banach spaces should be given to J. Bourgain [2].

We now wish to discuss the arrangement and the contents of this paper in greater detail.

In Section 1 we prove several results on subspaces of spaces of the form $X \oplus l_2$, X being a Banach space. These results, which are mostly infinite dimensional in nature, are essential for the construction of the space X. However, they are quite general and may have applications elsewhere.

Section 2 contains the major local results needed for our construction. We modify one of the key results of [9] for our purposes and extend it to a class of norms on $\mathbf{R}^n$ larger than those considered in [9]. These results are the building stones of our construction.

Section 3 contains the main results of the paper and the actual construction of a space having the properties mentioned in the title of the paper (Theorem 3.1). The section contains two other results, Theorems 3.3 and 3.4, which combined give the conclusion of Theorem 3.1.

The Sections 4 and 5 are devoted to the proof of Theorem 3.3 and Theorem 3.4, respectively.

0. Notation and terminology

In this paper we shall use the notation and terminology commonly used in Banach space theory, as it appears in [6] and [7]. If (g_n) is a sequence of independent Gaussian variables on a probability space $(\Omega, \mathscr{S}, \mu)$, each with distribution $N(0, 1)$ and X is a Banach space, we say that X is of type p, $1 \leqq p \leqq 2$, respectively cotype q, $2 \leqq q \leqq \infty$, if there is a constant $k \geqq 1$ so that

$$(1) \qquad \left(\int \left\| \sum_{j=1}^{n} g_j(t) x_j \right\|^2 d\mu(t) \right)^{1/2} \leqq k \left(\sum_{j=1}^{n} \| x_j \|^p \right)^{1/p},$$

respectively,

$$(2) \qquad \left(\int \left\| \sum_{j=1}^{n} g_j(t) x_j \right\|^2 d\mu(t) \right)^{1/2} \geqq k^{-1} \left(\sum_{j=1}^{n} \| x_j \|^q \right)^{1/q}$$

holds for all finite sets $\{x_1, x_2, \ldots, x_n\} \subseteq X$. The smallest constant which can be used in (1) is called the type p constant of X and denoted by $T_p(X)$ while the smallest constant which can be used in (2) is called the cotype q constant of X and denoted by $C_q(X)$.

If X is an n-dimensional Banach space, we shall identify X with $\mathbf{R}^n$, equipped with a suitable norm and write $X = (\mathbf{R}^n, \|\cdot\|_X)$. For $1 \leqq p \leqq \infty$ we let $\|\cdot\|_p$ denote the norm of l_p.

If $n \in \mathrm{N}$, $L(\mathbf{R}^n)$ is the space of all linear operators from $\mathbf{R}^n$ to $\mathbf{R}^n$. If $\|\cdot\|_X$ and $\|\cdot\|_Y$ are two norms on $\mathbf{R}^n$, $X = (\mathbf{R}^n, \|\cdot\|_X)$, $Y = (\mathbf{R}^n, \|\cdot\|_Y)$ and $T \in L(\mathbf{R}^n)$ the expression $\|T : X \to Y\|$ denotes the operator norm of T considered as an operator from X to Y, $\|T\|_X = \|T : X \to X\|$. Similarly $\gamma_2(T : X \to Y)$ denotes the l_2-factorization norm of T considered as an operator from $X \to Y$, i.e.

$$\gamma_2(T : X \to Y) = \inf\{\|A\| \, \|B\| \mid A : X \to l_2, B : l_2 \to Y, BA = T\}.$$

We put $\gamma_2(T, X) = \gamma_2(T : X \to X)$ and when it is clear from the context what X and Y are, we shall just write $\gamma_2(T)$.

$d(E, F)$ denotes the Banach–Mazur distance between the Banach spaces E and F, and if $\dim E = n$ we put $d(E) = d(E, l_2^n)$.

If (x_n) is a basis for a Banach space X, then there is a constant M so that

$$\left\| \sum_{j=1}^{n} t_j x_j \right\| \leqq M \left\| \sum_{j=1}^{m} t_j x_j \right\| \quad \text{for all } (t_j) \subseteq \mathbf{R} \text{ and all } n, m \in \mathrm{N}, \quad n < m.$$

The smallest constant which can be used in this inequality is called the basis constant of (x_n).

If X is a Banach space with a basis, we put

$$b(X) = \inf\{K \mid X \text{ has a basis with constant } K\}$$

and put $b(X) = \infty$ if X has no basis.

Let us end this section with the following definition:

0.1. Definition. Let $n \in \mathrm{N}$ and $\lambda \geqq 1$. A Banach space X is called (λ, n)-Euclidean if for every n-dimensional subspace $E \subseteq X$, we have $d(E) \leqq \lambda$.

1. Subspaces of $X \oplus l_2$ for certain Banach spaces X and other general results

In this section we prove some general results, especially on subspaces of $X \oplus l_2$, X being a Banach space. These results are crucial for the construction of our example, but they probably have applications elsewhere too.

The following lemma, which is a generalization of Maurey's extension property [10], easily follows from [1] and [15].

1.1. LEMMA. *Let X be a Banach space of type p, $1 \leqq p \leqq 2$ and cotype q, $2 \leqq q < \infty$. If $E \subseteq X$ is an n-dimensional subspace, then there is a projection P of X onto E with*

$$\gamma_2(P) \leqq 2T_p(X)C_q(X)n^{1/p-1/q}.$$

PROOF. Let $J : E \to E$ be the identity operator. By Benyamini and Gordon [1], Theorem 5.2 or Tomczak-Jaegermann [15], Theorem 25.10, J has an extension $P : X \to E$ with

$$\gamma_2(P) \leqq 2T_p(X)C_q(E)n^{1/p-1/q} \leqq 2T_p(X)C_q(X)n^{1/p-1/q}.$$

P is clearly the desired projection. ∎

Using this lemma, we obtain

1.2. PROPOSITION. *Let X be a reflexive Banach space of type p, $1 < p \leqq 2$ and cotype $q \geqq 2$. If $F \subseteq X$ is a subspace of codimension n, then there is a projection P of X onto F with*

$$\| P \| \leqq 2T_p(X)C_q(X)k(X)n^{1/p-1/q} + 1$$

where $k(X)$ is the K-convexity constant of X.

PROOF. It follows from [12] that X is K-convex and hence X^* is of type q' with $T_{q'}(X) \leqq k(X)C_q(X)$, $q^{-1} + q'^{-1} = 1$, and of cotype p', $p^{-1} + p'^{-1} = 1$. By Lemma 1.1 there is a projection Q of X^* onto $F^\perp$ with

$$\| Q \| \leqq \gamma_2(Q) \leqq 2T_p(X)C_q(X)n^{1/p-1/q}. \tag{1}$$

If φ denotes the quotient map of X onto X/F, $Q^*\varphi$ is a projection of X onto $Q^*\varphi(X)$. Since $\dim Q^*\varphi(X) = n$ and $Q^*\varphi$ is zero on F, it follows that $(Q^*\varphi)^{-1}(0) = F$.

The desired projection is $P = I - Q^*\varphi$. ∎

1.3. PROPOSITION. *Let $n \in \mathbf{N}$ and let $\| \cdot \|$ be a norm on $\mathbf{R}^n$ so that $\| x \| \leqq \| x \|_2$ for all $x \in \mathbf{R}^n$. Put $X = (\mathbf{R}^n, \| \cdot \|)$.*

If $Y \subseteq X \oplus_2 l_2^n$ is a subspace with $\dim Y = m \leqq n$, *then there is an operator $T : Y \to X$ so that*

$$\frac{1}{\sqrt{2}} \| Tx \| \leqq \| x \|_Y \leqq \| Tx \|_2 \quad \text{for all } x \in Y. \tag{i}$$

PROOF. Let P be the restriction to Y of the natural projection of $X \oplus_2 l_2^n$ onto X. Since $m \leqq n$ it follows from the spectral theorem for compact operators, see e.g. [5], that there is an orthonormal basis $\{e_i \mid 1 \leqq i \leqq m\}$ for $(Y, \|\cdot\|_2)$, an orthonormal sequence $\{f_i \mid 1 \leqq i \leqq m\} \subseteq (X, \|\cdot\|_2)$ and $0 \leqq \lambda_i \leqq 1$ for $i = 1, 2, \ldots, m$ so that

$$Px = \sum_{i=1}^{m} \lambda_i (x, e_i) f_i \quad \text{for all } x \in Y. \tag{1}$$

We define $T: Y \to X$ by $Te_i = f_i$ for all $1 \leqq i \leqq m$. It follows immediately that for all $x \in Y$ we have

$$\| x \|_Y \leqq \| x \|_2 = \| Tx \|_2 \tag{2}$$

which gives the right inequality of (i).

To prove the left inequality, we note that if $x \in Y$, then

$$x = Px + (x - Px) = Px + \sum_{i=1}^{m} (x, e_i)[e_i - \lambda_i f_i]. \tag{3}$$

By observing that $(e_i - \lambda_i f_i)$ is an orthogonal sequence in l_2^n with $\| e_i - \lambda_i f_i \|_2 = (1 - \lambda_i^2)^{1/2}$ we get from (3) that, for all $x \in Y$,

$$\| x \|_Y^2 = \| Px \|^2 + \sum_{i=1}^{m} |(x, e_i)|^2 (1 - \lambda_i^2). \tag{4}$$

Further, for all $x \in Y$:

$$\begin{aligned} \| Tx \| &\leqq \| Px \| + \left\| \sum_{i=1}^{m} (1 - \lambda_i)(x, e_i) f_i \right\| \\ &\leqq \sqrt{2} \left(\| Px \|^2 + \left\| \sum_{i=1}^{m} (1 - \lambda_i)(x, e_i) f_i \right\|_2^2 \right)^{1/2} \\ &\leqq \sqrt{2} \left(\| Px \|^2 + \sum_{i=1}^{m} (1 - \lambda_i)^2 |(x, e_i)|^2 \right)^{1/2} \\ &\leqq \sqrt{2} \| x \|_Y. \end{aligned} \tag{5}$$

The last inequality follows from (4) by observing that for $0 \leqq \lambda \leqq 1$ we have $(1 - \lambda)^2 \leqq 1 - \lambda^2$. ■

We also need:

1.4. Proposition. *Let X be an n-dimensional Banach space and $E \subseteq X \oplus_2 l_2$ a subspace. Then there is a subspace $F \subseteq X \oplus_2 l_2^n$ so that* $\dim F \leqq n$ *and a Hilbert space H so that E is 2-isomorphic to $F \oplus_2 H$.*

Proof. Let $P: X \oplus_2 l_2 \to l_2$ be the natural projection and put $H = E \cap l_2$. Clearly $P(E) \supseteq H$ and the codimension of H in $P(E)$ is at most n. Let Q be the orthogonal projection of $P(E)$ onto H and put $Q_1 = QP_{|E}$, $F = Q_1^{-1}(0)$. Q_1 is clearly a projection of E onto H and since $P(F) \subseteq Q^{-1}(0)$, $\dim P(F) \leqq n$. Hence F is isometric to a subspace of $X \oplus_2 l_2^n$. Since $E = F \oplus H$, the conclusion follows. ∎

Our final result of this section is a combination of Propositions 1.3 and 1.4:

1.5. Corollary. *Let $n \in \mathbf{N}$ and let $\|\cdot\|$ be a norm on $\mathbf{R}^n$ with $\|x\| \leqq \|x\|_2$ for all $x \in \mathbf{R}^n$, and put $X = (\mathbf{R}^n, \|\cdot\|)$.*

If $E \subseteq X \oplus_2 l_2$ is an infinite dimensional subspace then there is an n-dimensional subspace E_n of $X \oplus_2 l_2$ and an operator S of E_n onto X so that

(i) *E is 2-isomorphic to $E_n \oplus_2 l_2$,*

(ii) $\frac{1}{2}\|Sx\| \leqq \|x\|_{E_n} \leqq \|Sx\|_2$ *for all $x \in E_n$.*

Proof. Since E is infinite dimensional, it follows from Proposition 1.4 that there is a subspace $F \subseteq X \oplus_2 l_2^n$, $\dim F = m \leqq n$, so that E is 2-isomorphic to $F \oplus_2 l_2$. By Proposition 1.3 there is an operator $T: F \to X$ so that

$$\frac{1}{\sqrt{2}}\|Tx\| \leqq \|x\|_F \leqq \|Tx\|_2 \quad \text{for all } x \in F. \tag{1}$$

Let Y be the orthogonal complement to $T(F)$ in $(X, \|\cdot\|_2)$ and let $U: l_2^{n-m} \to (Y, \|\cdot\|_2)$ be a unitary operator. Define $E_n = F \oplus_2 l_2^{n-m}$. Clearly E is 2-isomorphic to $E_n \oplus_2 l_2$ and if we define $S: E_n \to X$ by $S = T \oplus U$, we get that if $x \in E_n$, $x = y + z$, $y \in F$, $z \in l_2^{n-m}$, then by (1),

$$\begin{aligned} \|Sx\| &\leqq \|Ty\| + \|Uz\|_2 \leqq \sqrt{2}\|y\|_F + \|z\|_2 \\ &\leqq \sqrt{2}(2\|y\|_F^2 + 2\|z\|_2^2)^{1/2} = 2\|x\|_{E_n}. \end{aligned} \tag{2}$$

It is obvious that S satisfies the other inequality in (ii). ∎

2. Some local results

In this section we present the local results which are essential for our construction. We start with the following theorem which is the key to what follows in the sequel, and which is a slight modification of Proposition 3.2 in [9].

2.1. THEOREM. *There exists a constant* $C \geqq 1$ *so that for every* $q \in]2, \infty]$ *and every* $n \in \mathbf{N}$, $n \geqq 2$ *there is a norm* $\|\cdot\|_{n,q}$ *on* $\mathbf{R}^n$, *so that if* $X_q^n = (\mathbf{R}^n, \|\cdot\|_{n,q})$ *then the following conditions are satisfied*:

(i) $\|x\|_{n,q} \leqq \|x\|_2$ *for all* $x \in \mathbf{R}^n$.

(ii) X_q^n *is isometric to a subspace of* l_q^{2n}.

(iii) *For every* $T \in L(\mathbf{R}^n)$ *there is a* $\lambda_T \in \mathbf{R}$, *a* $V_T \in L(\mathbf{R}^n)$ *and a subspace* $E_T \subseteq \mathbf{R}^n$, $\dim E_T \geqq 31n/32$ *so that for all* $q \in]2, \infty]$

  (a) $T = \lambda_T I + V_T$,

  (b) $\|V_{|E_T}\|_2 \leqq C n^{1/q - 1/2} \|T\|_{X_q^n}$,

  (c) $|\lambda_T| \leqq C \|T\|_{X_q^n}$.

(iv) *For every* $T \in L(\mathbf{R}^n)$ *there is a subspace* $F_T \subseteq \mathbf{R}^n$, $\dim E_T \geqq 31n/32$ *so that for all* $q \in]2, \infty]$:

$$\|T_{|F}\|_2 \leqq C n^{1/q - 1/2} \gamma_2(T, X_q^n).$$

(v) *For every* $T \in L(\mathbf{R}^n)$ *there is a subspace* G_T *with* $\dim G_T \geqq 31n/32$, *so that for all* $q \in]2, \infty]$:

$$\|T_{|G}\|_2 \leqq C \|T : l_2^n \to X_q^n\|.$$

PROOF. The existence of the norms $\|\cdot\|_{n,q}$ satisfying (i)–(iv) can be shown in exactly the same way as in [9], Proposition 3.2 (the replacement of $7n/8$ by $31n/32$ can be done by a suitable modification of Lemma 2.1 in [8]). As a by-product of the proof of (iii) we get that the norms $\|\cdot\|_{n,q}$ also satisfy (v) (cf. the proof of Proposition 3.1 in [8]). ∎

We now wish to define a class of spaces which have similar properties as the spaces X_q^n of Theorem 2.1.

2.2. DEFINITION. Let $q \in]2, \infty]$, $n \in \mathbf{N}$. If $\|\cdot\|$ is a norm on $\mathbf{R}^n$ and $Y = (\mathbf{R}^n, \|\cdot\|)$ then we shall say that $Y \in SN(n, q)$ if $\frac{1}{2}\|x\|_{n,q} \leqq \|x\| \leqq \|x\|_2$ for all $x \in \mathbf{R}^n$.

2.3. DEFINITION. Let $n \in \mathbf{N}$, $q \in]2, \infty]$ and $Y \in SN(n, q)$. Put

$$\Lambda(n, q, Y) = \{\lambda_T \in \mathbf{R} \mid T \in L(\mathbf{R}^n),\ \|T : X_q^n \to Y\| \leqq 1\}$$

where, for each $T \in L(\mathbf{R}^n)$, λ_T is a real number satisfying the conclusions of Theorem 2.1.

By definition it follows that if $Y \in SN(n, q)$, then

$$\text{(A)} \qquad \| T \|_{X_q^n} \leqq 2 \| T : X_q^n \to Y \| \quad \text{for all } T \in L(\mathbf{R}^n)$$

and hence from Theorem 2.1 we infer that the set $\Lambda(n, q, Y)$ is bounded. We put, for every $n \in \mathbf{N}$, $q \in]2, \infty]$ and $Y \in SN(n, q)$,

$$\lambda(n, q, Y) = \sup \Lambda(n, q, Y).$$

The next proposition and its corollary describe some crucial properties of the spaces in $SN(n, q)$.

2.4. Proposition. *Let C be the constant of Theorem 2.1, let $n \in \mathbf{N}$, $n \geqq 2$ and $q \in]2, \infty]$. If $Y \in SN(n, q)$ then we have*

(i) *For every $T \in L(\mathbf{R}^n)$ there is a $v_T \in \mathbf{R}$, a $\tilde{V}_T \in L(\mathbf{R}^n)$, and a subspace $\tilde{E}_T \subseteq \mathbf{R}^n$ with $\dim \tilde{E}_T \geqq 7n/8$ so that*

(a) $T = v_T I + \tilde{V}_T$,

(b) $| v_T | \leqq 2C \| T \|_Y \lambda(n, q, Y)^{-1}$,

(c) $\| V_{T|\tilde{E}_T} \|_2 \leqq 6C^2 \| T \|_Y n^{1/q - 1/2} \lambda(n, q, Y)^{-1}$.

(ii) *For every $T \in L(\mathbf{R}^n)$ there is a subspace $\tilde{F}_T \subseteq \mathbf{R}^n$ with $\dim \tilde{F}_T \geqq 7n/8$ so that*

$$\| T_{|\tilde{F}_T} \|_2 \leqq 6C^2 n^{1/q - 1/2} \gamma_2(T, Y) \lambda(n, q, Y)^{-1}.$$

(iii) *For every $T \in L(\mathbf{R}^n)$ there is a subspace $\tilde{G}_T \subseteq \mathbf{R}^n$ with $\dim \tilde{G}_T \geqq 31n/32$ so that*

$$\| T_{|\tilde{G}_T} \|_2 \leqq 2C \| T : l_2^n \to Y \| .$$

Proof. Let $n \in \mathbf{N}$, $n \geqq 2$, $q \in]2, \infty]$, $Y \in SN(n, q)$ and $T \in L(\mathbf{R}^n)$ be given.

(iii) Follows easily from (v) of Theorem 2.1 by observing that the definition of $SN(n, q)$ implies that the subspace G_T from there can be used as $\tilde{G}_T$.

To prove (i), set $\lambda_0 = \lambda(n, q, Y)$. An easy compactness argument yields that there is a $T_0 \in L(\mathbf{R}^n)$ with $\| T_0 : X_q^n \to Y \| = 1$ so that Theorem 2.1 is satisfied with $\lambda_{T_0} = \lambda_0$. We now define

$$(1) \qquad v_T = \lambda_{TT_0} \lambda_0^{-1}.$$

By (A) and Theorem 2.1 (iii)(c) we get

$$
\begin{aligned}
|\nu_T| &\leqq |\lambda_{TT_0}|\lambda_0^{-1} \leqq C\lambda_0^{-1} \| TT_0 \|_{X_q^n} \\
&\leqq 2C\lambda_0^{-1} \| TT_0 : X_q^n \to Y \| \qquad (2)\\
&\leqq 2C\lambda_0^{-1} \| T \|_Y \| T_0 : X_q^n \to Y \| \leqq 2C\lambda_0^{-1} \| T \|_Y
\end{aligned}
$$

which proves (i), (b).

Since

$$T(\lambda_0 I + V_{T_0}) = TT_0 = \lambda_{TT_0} I + V_{TT_0} \tag{3}$$

we obtain from (1) and (3)

$$T = \nu_T I + \lambda_0^{-1}(V_{TT_0} - TV_{T_0}). \tag{4}$$

We now define

$$\tilde{V}_T = \lambda_0^{-1}(V_{TT_0} - TV_{T_0}), \tag{5}$$

$$\tilde{E}_T = E_{TT_0} \cap E_{T_0} \cap \tilde{G}_T. \tag{6}$$

By (A) and Theorem 2.1 we get

$$
\begin{aligned}
\| \tilde{V}_{T|\tilde{E}_T} \|_2 &\leqq \lambda_0^{-1}(\| V_{TT_0|E_{TT_0}} \|_2 + \| T_{\tilde{G}_T} \|_2 \| V_{T_0|E_{T_0}} \|_2) \\
&\leqq \lambda_0^{-1} n^{1/q-1/2}(C \| TT_0 \|_{X_q^n} + 2C^2 \| T : l_2^n \to Y \| \; \| T_0 \|_{X_q^n}) \qquad (7)\\
&\leqq \lambda_0^{-1} n^{1/q-1/2}(2C \| T \|_Y + 4C^2 \| T \|_Y) \\
&\leqq 6C^2 \lambda_0^{-1} n^{1/q-1/2} \| T \|_Y
\end{aligned}
$$

where we have used (A) and that $\| T : l_2^n \to Y \| \leqq \| T \|_Y$.

Finally, it is immediate that $\dim \tilde{E}_T \geqq 7n/8$. This concludes the proof of (i).

To prove (ii), we first observe that

$$\gamma_2(TT_0, X_q^n) \leqq 2\gamma_2(T, Y) \quad \text{for all } T \in L(\mathbf{R}^n). \tag{8}$$

Define

$$\tilde{F}_T = F_{TT_0} \cap E_{T_0} \cap \tilde{G}_T. \tag{9}$$

By isolating $\lambda_0 T$ from the first equality in (3), we infer that

$$
\begin{aligned}
\lambda_0 \| T_{|\tilde{F}_T} \|_2 &\leqq \| TT_{0_{|F_{TT_0}}} \|_2 + \| T_{|\tilde{G}_T} \|_2 \| V_{T_0|E_{T_0}} \|_2 \\
&\leqq Cn^{1/q-1/2}\gamma_2(TT_0, X_q^n) + 4C^2 n^{1/q-1/2} \| T \|_Y \qquad (10)\\
&\leqq 6C^2 n^{1/q-1/2}\gamma_2(T, Y)
\end{aligned}
$$

and that $\dim \tilde{F}_T \geqq 7n/8$.

This proves (ii). ■

From Theorem 2.1 and Proposition 2.4 we get

2.5. COROLLARY. *If $n \in \mathbf{N}$, $n \geqq 2$, $q \in]2, \infty]$ and $Y \in SN(n, q)$, then we have:*

(i) *For all $T \in L(\mathbf{R}^n)$,*

$$|\nu_T - 1| \leqq \| T - I \|_2 + 6C^2 \lambda(n, q, Y)^{-1} n^{1/q - 1/2} \| T \|_Y,$$

$$|\lambda_T - 1| \leqq \| T - I \|_2 + C \| T \|_{X_q^n} n^{1/q - 1/2}.$$

(ii) *For all $T_1, T_2 \in L(\mathbf{R}^n)$*, $\mathrm{rk}(T_1 - T_2) \leqq n/2$:

$$|\nu_{T_1} - \nu_{T_2}| \leqq 6C^2 (\| T_1 \|_Y + \| T_2 \|_Y) \lambda(n, q, Y)^{-1} n^{1/q - 1/2},$$

$$|\lambda_{T_1} - \lambda_{T_2}| \leqq C (\| T_1 \|_{X_q^n} + \| T_2 \|_{X_q^n}) n^{1/q - 1/2}.$$

PROOF. By Proposition 2.4(i), we get for every $x \in \tilde{E}_T$, $\| x \|_2 = 1$:

$$\begin{aligned} \| T - I \|_2 &\geqq \| Tx - x \|_2 = \| (\nu_T - 1)x + \tilde{V}_T x \|_2 \\ &\geqq |\nu_T - 1| - 6C^2 \| T \|_Y \lambda(n, q, Y)^{-1} n^{1/q - 1/2} \end{aligned} \tag{1}$$

which yields the first estimate in (i).

The second one can be proved in exactly the same way by using Theorem 2.1(ii).

To prove (ii), we note that since $\mathrm{rk}(T_1 - T_2) \leqq n/2$, $Z = (T_1 - T_2)^{-1}(0) \cap \tilde{E}_{T_1} \cap \tilde{E}_{T_2} \neq \{0\}$. If $x \in B_Z$ we have

$$\begin{aligned} 0 &= \| T_1 x - T_2 x \|_2 = \| (\nu_{T_1} x - \nu_{T_2} x) + (\tilde{V}_{T_1} x - \tilde{V}_{T_2} x) \|_2 \\ &\geqq |\nu_{T_1} - \nu_{T_2}| - \| \tilde{V}_{T_1} x \|_2 - \| \tilde{V}_{T_2} x \|_2. \end{aligned} \tag{2}$$

The first inequality in (ii) now follows from Proposition 2.4(c). The second one follows from Theorem 2.1 in the same manner. ■

3. The main results

In this section we shall construct the Banach space X with the properties mentioned in the introduction and state our main results.

Inductively we define sequences $(n_k) \subseteq \mathbf{N}$ and $(q_k) \subseteq]2, \infty]$ so that

$$n_1 = q_1 = 4, \tag{B}$$

(C) $$(2n_{k-1})^{1/2-1/q_k} \leqq 2,$$

(D) $$n_k \geqq \sum_{i=1}^{k-1} n_i \quad \text{and} \quad n_k^{1/4-1/2q_k} n_{k-1}^{-1} \geqq k,$$

for all $k \geqq 2$.

The space X is now defined as

(E) $$X = \left(\sum_{n=1}^{\infty} \oplus X_{q_k}^{n_k} \right)_2 .$$

Our main results on X are:

3.1. THEOREM. *Let $Y \subseteq X \oplus X$ be a subspace, so that $X \oplus \{0\} \subseteq Y$. Then Y does not have a basis.*

Since every Y with the properties of Theorem 3.1 is of the form $X \oplus E$ for a suitable subspace $E \subseteq X$, Theorem 3.1 can be reformulated to

3.2. THEOREM. *For no subspace $E \subseteq X$ does $X \oplus E$ have a basis.*

Theorem 3.2, and hence 3.1, will follow from the following two theorems where the first one will be proved in Section 4 and the second one in Section 5.

3.3. THEOREM. *Let $E \subseteq X$ be a subspace. For every $k \in \mathbf{N}$ there is a Banach space $Y \in SN(n_k, q_k)$ and a $(16, n_k)$-Euclidean Banach space Z so that $X \oplus_2 E$ is $1152 n_{k-1}^{1/2}$-isomorphic to $X_{q_k}^{n_k} \oplus_2 Y \oplus_2 Z$.*

3.4. THEOREM. *There is a $\delta > 0$ so that if $n \in \mathbf{N}$, $n \geqq 2$, $q \in]2, \infty]$, $\mu \geqq 1$, $Y \in SN(n, q)$ and Z is a (μ, n)-Euclidean Banach space, then*

$$b = b(X_q^n \oplus_2 Y \oplus_2 Z) \geqq \delta \mu^{-1/2-1/2q}.$$

PROOF OF THEOREM 3.2. If $E \subseteq X$ is a subspace then, by Theorems 3.3 and 3.4, we have for every $k \in \mathbf{N}$:

$$b(X \oplus_2 E) \geqq \frac{1}{4608} \delta n_{k-1}^{-1/2} n_k^{1/4-1/2q_k} \geqq \frac{\delta}{4608} k$$

and hence $b(X \oplus_2 E) = \infty$. ∎

4. Proof of Theorem 3.3

The essential step of the proof is formulated in the next proposition.

4.1. PROPOSITION. *Let $E \subseteq X$ be a subspace and $k \in \mathbf{N}$. There exist a $\tilde{Y} \in SN(n_k, q_k)$ and a $(16, n_k)$-Euclidean Banach space $\tilde{Z}$ so that $d(E \oplus_2 l_2, \tilde{Y} \oplus_2 \tilde{Z}) \leqq 288\sqrt{2}n_{k-1}^{1/2}$.*

PROOF. Put

$$Y_0 = \left(\sum_{i=1}^{k} X_{q_i}^{n_i}\right)_2; \quad Y_1 = \left(\sum_{i=k+1}^{\infty} X_{q_i}^{n_i}\right)_2, \quad E_1 = E \cap Y_1. \tag{1}$$

We denote by P the restriction to E of the natural projection of X onto Y_1; clearly $E_1 \subseteq P(E)$.

By the choice of the n_i's, $\dim Y_0 \leqq 2n_k$, and therefore the codimension of E_1 in $P(E)$ is not greater than $2n_k$. Hence, by Proposition 1.2 there is a projection Q of $P(E)$ onto E_1 so that

$$\| Q \| \leqq 2 \cdot 4(2n_k)^{1/2 - 1/q_{k+1}} + 1 \leqq 17 \tag{2}$$

(note that $T_2(Y_1)C_4(Y_1) \leqq 4$ and $k(X) = 1$).

Let $Q_1 = QP$ and put $E_0 = Q_1^{-1}(0)$. Clearly Q_1 is a projection of E onto E_1, and since $P(E_0) \subseteq Q^{-1}(0)$, we get that $\dim P(E_0) \leqq 2n_k$. Hence, by [15], Corollary 25.11:

$$d(P(E_0)) \leqq 2 \cdot 4 \cdot (2n_k)^{1/2 - 1/q_{k+1}} \leqq 16. \tag{3}$$

Now (2) gives that

$$d(E, E_0 \oplus_2 E_1) \leqq 18\sqrt{2}. \tag{4}$$

From the fact that

$$F_1 = E_0 \oplus_2 l_2 \subseteq P(E_0) \oplus_2 \left(\sum_{i=1}^{k-1} X_{q_i}^{n_i}\right)_2 \oplus_2 X_{q_k}^{n_k} \oplus_2 l_2$$

and $\dim(\sum_{i=1}^{k-1} X_{q_i}^{n_i})_2 \leqq 2n_{k-1}$, we get using (3) that there is a subspace $F \subseteq X_{q_k}^{n_k} \oplus_2 l_2$ so that

$$d(F_1, F) \leqq \max(\sqrt{2}n_{k-1}^{1/2}, 16) \leqq 8n_{k-1}^{1/2}. \tag{5}$$

From Corollary 1.5 we infer that there is an n_k-dimensional subspace $F_{n_k} \subseteq X_{q_k}^{n_k} \oplus_2 l_2$ and an operator $S : F_{n_k} \to X_{q_k}^{n_k}$ satisfying (i) and (ii) there. Let

$|||\cdot|||$ be the norm on $\mathbf{R}^{n_k}$ having $S(B_{F_{n_k}})$ as its unit ball and put $\tilde{Y}=(\mathbf{R}^{n_k},|||\cdot|||)$.

It follows immediately that $\tilde{Y}\in SN(n_k,q_k)$ and that

$$(6)\qquad d(F_1,\tilde{Y}\oplus_2 l_2)\leqq d(F_1,F)d(F,\tilde{Y}\oplus_2 l_2)\leqq 16n_{k-1}^{1/2}.$$

Let us now define $\tilde{Z}=E_1\oplus_2 l_2$. By the choice of n_k and q_{k+1} it follows that $\tilde{Z}$ is $(16,n_k)$-Euclidean (cf. again [15], Corollary 25.11).

From (4) and (6) we obtain:

$$(7)\qquad d(E\oplus_2 l_2,\tilde{Y}\oplus_2\tilde{Z})\leqq 288\sqrt{2}n_{k-1}^{1/2}.\qquad\blacksquare$$

Though we do not really need the next corollary in the proof, it may be of some interest.

4.2. Corollary. *If $E\subseteq X$ is an infinite dimensional subspace and $k\in\mathbf{N}$, then there exists a $Y_E\in SN(n_k,q_k)$ and a $(16,n_k)$-Euclidean Banach space Z_E so that $d(E,Y_E\oplus_2 Z_E)\leqq cn_{k-1}^{1/2}$ for some universal constant c.*

Proof. By a standard gliding hump argument it follows that E contains a subspace H with $d(H,l_2)\leqq 3/2$. Since X is of type 2 with $T_2(X)\leqq 4/3$ it follows from [10] that there is a projection P of E onto H with $\|P\|\leqq 4/3\cdot 3/2=2$. Hence, by the decomposition method $d(E\oplus_2 l_2,E)\leqq 27$ and the conclusion follows from Proposition 4.1. $\blacksquare$

Proof of Theorem 3.3. Let $E\subseteq X$ be a subspace, let $k\in\mathbf{N}$ and let $\tilde{Y}$ and $\tilde{Z}$ be as in Proposition 4.1.

Since $(\Sigma_{i=k+1}^{\infty}X_{q_i}^{n_i})_2$ contains a 1-complemented isometric copy of l_2 it contains a subspace Y_2 so that

$$(1)\qquad d\left(\left(\sum_{j=k+1}^{\infty}X_{q_i}^{n_i}\right)_2,Y_2\oplus_2 l_2\right)\leqq 2\sqrt{2}.$$

Let $Y=\tilde{Y}$ and $Z=\tilde{Z}\oplus_2 Y_2\oplus_2 l_2$. Clearly $Y\in SN(n_k,q_k)$ and Z is $(16,n_k)$-Euclidean. Since $\dim(\Sigma_{i=1}^{k-1}X_{q_i}^{n_i})\leqq 2n_{k-1}$ it easily follows from Proposition 4.1 and (1) that

$$(2)\qquad d(X\oplus_2 E,X_{q_k}^{n_k}\oplus_2 Y\oplus_2 Z)\leqq 1152n_{k-1}^{1/2}.\qquad\blacksquare$$

5. Proof of Theorem 3.4

Let $n \in \mathbf{N}$, $n \geqq 2$, $2 < q < \infty$, $\mu \geqq 1$, $Y \in SN(n, q)$ and Z a (μ, n)-Euclidean Banach space be given. We assume that $X_q^n \oplus_2 Y \oplus_2 Z$ has a basis (y_n) with basis constant b.

For every m we let P_m denote the mth partial sum projection for the basis (y_n) and we write P_m in the form of a matrix with operator entries, that is

$$P_m = \begin{Bmatrix} A_{11}^m & A_{12}^m & A_{13}^m \\ A_{21}^m & A_{22}^m & A_{23}^m \\ A_{31}^m & A_{32}^m & A_{33}^m \end{Bmatrix} \begin{matrix} X_q^n \\ Y \\ Z \end{matrix} \tag{1}$$

By Theorem 2.1 and Proposition 2.4 we can write, for every $m \in \mathbf{N}$,

$$\begin{aligned} A_{11}^m &= \lambda_{A_{11}^m} I + V_{A_{11}^m}, \\ A_{21}^m &= \lambda_{A_{21}^m} I + V_{A_{21}^m}, \\ A_{22}^m &= \nu_{A_{22}^m} I + \tilde{V}_{A_{22}^m}. \end{aligned} \tag{2}$$

There are now two cases to consider:
(I) $C^2 b n^{1/2q - 1/4} \geqq 1/64$,
(II) $C^2 b n^{1/2q - 1/4} < 1/64$,
where C is the constant from Theorem 2.1.

If (I) holds, then clearly

$$b \geqq \frac{1}{64} \cdot \frac{1}{C^2} n^{1/4 - 1/2q} \tag{3}$$

which is the conclusion of Theorem 3.4.

Case II. Since $P_m x \to x$ for all $x \in X_q^n \oplus_2 Y \oplus_2 Z$, we get that $A_{11}^m \to I$ and $A_{22}^m \to I$ for $m \to \infty$ and therefore, by Corollary 2.5(i), we get

$$\liminf |\lambda_{A_{11}^m}| \geqq 1 - bCn^{1/q - 1/2} \geqq 1 - \frac{1}{b} \cdot \left(\frac{1}{64}\right)^2 > \frac{63}{64}. \tag{4}$$

We now wish to prove the following

CLAIM. *There is an m_0 so that*

$$\text{either} \quad \tfrac{1}{4} \leqq |\lambda_{A_{11}^{m_0}}| \leqq \tfrac{3}{4} \quad \text{or} \quad \tfrac{1}{4} \leqq |\nu_{A_{22}^{m_0}}| \leqq \tfrac{3}{4} \tag{5}$$

and

$$|\lambda_{A_{21}^{m_0}}| \leqq 200b^2C^3b_k^{1/2q-1/4}\mu. \tag{6}$$

PROOF OF THE CLAIM. If $\lambda(n, q, Y) < n^{1/2q-1/4}$, then it follows from the definition of $\lambda(n, q, Y)$ that (6) holds for all $m \geqq 0$. By (4) there is a smallest number $m_0 \geqq 1$ so that $|\lambda_{A_{11}^{m_0}}| \geqq \frac{1}{4}$ and using Corollary 2.5(ii), we get

$$\begin{aligned} |\lambda_{A_{11}^{m_0}}| &\leqq |\lambda_{A_{11}^{m_0-1}}| + | |\lambda_{A_{11}^{m_0}} - |\lambda_{A_{11}^{m_0-1}}| \\ &\leqq \tfrac{1}{4} + 2Cbn^{1/q-1/2} \leqq \tfrac{1}{4} + \tfrac{1}{32} < \tfrac{3}{4}. \end{aligned} \tag{7}$$

Now assume that $\lambda(n, q, Y) \geqq n^{1/2q-1/4}$. By Corollary 2.5(i) we get

$$\liminf |\nu_{A_{22}^m}| \geqq 1 - 6C^2b\lambda(n, q, Y)^{-1}n^{1/q-1/2} \geqq 1 - 6C^2bn^{1/2q-1/4} > \tfrac{7}{8}. \tag{8}$$

(4) and (8) show that there is a smallest number m_0 so that

$$|\lambda_{A_{11}^{m_0}}| + |\nu_{A_{22}^{m_0}}| \geqq \tfrac{1}{2}. \tag{9}$$

Again Corollary 2.5(ii) shows that

$$|\lambda_{A_{11}^{m_0}}| + |\nu_{A_{22}^{m_0}}| \leqq \tfrac{1}{2} + 2bCn^{1/q-1/2} + 12C^2n^{1/2q-1/4} < \tfrac{1}{2} + \frac{1}{32} + \tfrac{3}{16} < \tfrac{3}{4}. \tag{10}$$

The choice of m_0 and (10) shows that (5) is satisfied.

In order to simplify the notation, we shall omit the index m_0 in the rest of the proof of Theorem 3.4.

To prove that the chosen m_0 will also satisfy (6), we first use $P^2 = P$ together with (1) to obtain

$$A_{21} = A_{21}A_{11} + A_{22}A_{21} + A_{23}A_{31} \tag{11}$$

which, together with (2), gives:

$$\begin{aligned} \lambda_{A_{21}}(1 - \lambda_{A_{11}} - \nu_{A_{22}})I = {} & V_{A_{21}} + \lambda_{A_{21}}V_{A_{11}} + \lambda_{A_{11}}V_{A_{21}} + V_{A_{21}}V_{A_{11}} \\ & + \lambda_{A_{21}}\tilde{V}_{A_{22}} + \nu_{A_{22}}V_{A_{21}} + \tilde{V}_{A_{22}}V_{A_{21}} + A_{23}A_{31}. \end{aligned} \tag{12}$$

Note that the operators A_{23} and A_{31} represent a factorization of $A_{23}A_{31}$ through the (μ, n)-Euclidean Banach space Z and therefore

$$\gamma_2(A_{23}A_{31}, X_q^n) \leqq \| A_{23} \| \, \| A_{31} \| \mu \leqq b^2\mu. \tag{13}$$

If we denote the operator on the left-hand side of (12) by L and the one on the right-hand side by R, we get from (10) that for all $x \in \mathbf{R}^n$:

$$\| Lx \|_2 \geqq \tfrac{1}{4}|\lambda_{A_{21}}| \, \| x \|_2 \tag{14}$$

while Proposition 2.4 and Theorem 2.1 give that for all

$$x \in E_{A_{21}} \cap E_{A_{11}} \cap V_{A_{11}}^{-1}(E_{A_{21}}) \cap \tilde{E}_{A_{22}} \cap V_{A_{21}}^{-1}(\tilde{E}_{A_{22}}) \cap F_{A_{23}A_{31}}$$

(which is not the zero space!), we have:

$$\| Rx \|_2 \leqq 50C^3 b^2 \mu n^{1/2q-1/4} \| x \|_2. \tag{15}$$

Now (14) and (15) give (6) and hence conclude the proof of the claim.

Using (5) in the claim, there are now two cases to consider; in the sequel we shall assume that $\frac{1}{4} \leqq |\lambda_{A_{11}}| \leqq \frac{3}{4}$, since the other case can be treated in exactly the same way.

Again, using $P^2 = P$ together with (1), we obtain

$$A_{11} = A_{11}A_{11} + A_{12}A_{21} + A_{13}A_{31} \tag{16}$$

which, together with (2), implies

$$(\lambda_{A_{11}} - \lambda_{A_{11}}^2)I = (2\lambda_{A_{11}} - 1)V_{A_{11}} + \lambda_{A_{21}}A_{12} + A_{12}V_{A_{21}} + A_{13}A_{31}. \tag{17}$$

Since Z is (μ, n)-Euclidean, we get that

$$\gamma_2(A_{13}A_{31}) \leqq b^2\mu. \tag{18}$$

Denoting the operator on the left-hand side of (17) by L_1 and the one on the right-hand side by R_1, we get that for all $x \in \mathbf{R}^n$

$$\| L_1 x \|_2 \geqq \tfrac{3}{16} \| x \|_2 \tag{19}$$

while Theorem 2.1, Proposition 2.4, and (6) give that for all

$$x \in E_{A_{11}} \cap V_{A_{11}}^{-1}(E_{A_{11}}) \cap E_{A_{21}} \cap V_{A_{21}}^{-1}(G_{A_{21}}) \cap F_{A_{13}A_{31}}$$

(which is not equal to $\{0\}$!)

$$\| R_1 x \|_2 \leqq 420C^4 b^2 n^{1/q-1/2}\mu \| x \|_2. \tag{20}$$

Finally, (19) and (20) give

$$b^2 \geqq \frac{1}{2240} \cdot \frac{1}{C^4} n^{1/2-1/q}\mu^{-1} \tag{21}$$

which is the desired estimate.

This finishes the proof of Theorem 3.4. ∎

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